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Block numbers of permutations and Schur-positivity

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Abstract. The *block number* of a permutation is the maximal number of components in its expression as a direct sum. We show that the distribution of the set of left-to-right-maxima over 321-avoiding permutations with a given block number k is equal to the distribution of this set over 321-avoiding permutations with the last descent of the inverse permutation at position $n-k$. This result is analogous to the Foata-Schützenberger equi-distribution theorem, and implies Schur-positivity of the quasi-symmetric generating function of descent set over 321-avoiding permutations with a prescribed block number.

Keywords: Schur positivity, permutation statistics, pattern avoidance, quasi-symmetric function.

1 Introduction

Given any subset A of the symmetric group $\mathcal{S}_n$, define the quasi-symmetric function

$$\mathcal{Q}(A) := \sum_{\pi \in A} \mathcal{F}_{n,\mathrm{Des}(\pi)},$$

where $\mathrm{Des}(\pi) := \{i : \ \pi(i) > \pi(i+1)\}$ is the *descent set* of π and $\mathcal{F}_{n,D}$ (for $D \subseteq [n-1]$) are Gessel's *fundamental quasi-symmetric functions;* see Section 2.3 for more details. The following long-standing problem was first posed in [10].

Problem 1.1. For which subsets $A \subseteq \mathcal{S}_n$ is $\mathcal{Q}(A)$ symmetric?

A symmetric function is *Schur-positive* if all the coefficients in its expansion in the basis of Schur functions are nonnegative. Determining whether a given symmetric function is Schur-positive is a major problem in contemporary algebraic combinatorics [16].

Call a subset $A \subseteq \mathcal{S}_n$ *Schur-positive* if $\mathcal{Q}(A)$ is symmetric and Schur-positive. Classical examples of Schur-positive sets of permutations include inverse descent classes [9],

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Knuth classes [9], conjugacy classes [10, Theorem 5.5], and permutations with a fixed inversion number [2, Prop. 9.5].

New constructions of Schur-positive sets of permutations were described in [8] and [13]. Inspired by these examples, Sagan and Woo raised the problem of finding Schur-positive pattern-avoiding sets [13].

The goal of this paper is to present a new example of a Schur-positive set of permutations which involves pattern-avoidance: the set of 321-avoiding permutations having a prescribed number of blocks. We shall state that more explicitly.

A permutation $\pi \in \mathcal{S}_n$ is 321-*avoiding* if the sequence $(\pi(1), \ldots, \pi(n))$ contains no decreasing subsequence of length 3. Denote by $\mathcal{S}_n(321)$ the set of 321-avoiding permutations in $\mathcal{S}_n$. For a permutation $\pi \in \mathcal{S}_n$ let

$$\mathrm{bl}(\pi) := |\{i : (\forall j \le i)\, \pi(j) \le i\}|$$

be the *block number* of π. The block number was studied in [17] and references therein, as the cardinality of the *connectivity set* of π. Denote

$$Bl_{n,k} := \{\pi \in \mathcal{S}_n(321) : \mathrm{bl}(\pi) = k\}.$$

Recall the *Frobenius characteristic map* ch, from class functions on $\mathcal{S}_n$ to symmetric functions, defined by $\mathrm{ch}(\chi^\lambda) = s_\lambda$ and extended by linearity. Our main result is:

Theorem 1.2. *For any $1 \le k \le n$, the set $Bl_{n,k}$ is Schur-positive. In fact, for $1 \le k \le n-1$*

$$\mathcal{Q}(Bl_{n,k}) = \mathrm{ch}(\chi^{(n-1,n-k)} \downarrow_{\mathcal{S}_n}^{\mathcal{S}_{2n-k-1}}),$$

where $\chi \downarrow_H^G$ stands for the restriction of the character χ from the group G to the group H; and, for $k = n$

$$\mathcal{Q}(Bl_{n,n}) = \mathrm{ch}(\chi^{(n)}) = s_{(n)}.$$

The coefficients of the Schur expansion of $\mathcal{Q}(Bl_{n,k})$ are described in Equation (5.1) below.

The proof of Theorem 1.2 involves a left-to-right-maxima-preserving bijection and a resulting equi-distribution result. Specifically, let

$$\mathrm{ltrMax}(\pi) := \{i : \pi(i) = \max\{\pi(1), \ldots, \pi(i)\}\}$$

be the set of *left-to-right maxima* in a permutation π. The *descent set* of π is

$$\mathrm{Des}(\pi) := \{i : \pi(i) > \pi(i+1)\}.$$

Define

$$\mathrm{ldes}(\pi) := \max\{i : i \in \mathrm{Des}(\pi)\}$$

to be the *last descent* of π, with $\mathrm{ldes}(\pi) := 0$ if $\mathrm{Des}(\pi) = \varnothing$ (i.e., if π is the identity permutation).

For every $I \subseteq [n]$, let $\mathbf{x}^I := \prod_{i \in I} x_i$. Our equi-distribution result is:

Theorem 1.3. *For every positive integer n*

$$\sum_{\pi\in\mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi)} q^{\mathrm{bl}(\pi)} = \sum_{\pi\in\mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi)} q^{n-\mathrm{ldes}(\pi^{-1})}.$$

See also Corollary 4.6 below.

Remark 1.4. *An equivalent formulation, replacing π by π^{-1} and using $\mathrm{bl}(\pi^{-1}) = \mathrm{bl}(\pi)$, is:*

$$\sum_{\pi\in\mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi^{-1})} q^{\mathrm{bl}(\pi)} = \sum_{\pi\in\mathcal{S}_n(321)} \mathbf{x}^{\mathrm{ltrMax}(\pi^{-1})} q^{n-\mathrm{ldes}(\pi)}.$$

It is reminiscent of the classical Foata-Schützenberger Theorem

$$\sum_{\pi\in\mathcal{S}_n} \mathbf{x}^{\mathrm{Des}(\pi^{-1})} q^{\mathrm{inv}(\pi)} = \sum_{\pi\in\mathcal{S}_n} \mathbf{x}^{\mathrm{Des}(\pi^{-1})} q^{\mathrm{maj}(\pi)};$$

see Observation 2.2 below.

After some necessary preliminaries in Section 2, we shall state a basic enumerative result in Section 3. Then Section 4 will outline the main idea in the proof of Theorem 1.3, and Section 5 will deduce Theorem 1.2 from a corollary of Theorem 1.3. Section 6 contains some final remarks.

2 Preliminaries

2.1 Statistics on permutations and on SYT

For a positive integer n let $[n] := \{1, 2, \ldots, n\}$, and let $\mathcal{S}_n$ denote the n-th symmetric group, the group of all permutations of $[n]$.

Observation 2.1. If $\pi \in \mathcal{S}_n$ then the restriction of π to the set $\mathrm{ltrMax}(\pi)$ is monotone increasing. If, moreover, $\pi \in \mathcal{S}_n(321)$ then the restriction of π to the complementary set $[n] \setminus \mathrm{ltrMax}(\pi)$ is also monotone increasing.

Observation 2.2. If $\pi \in \mathcal{S}_n(321)$ then the set $\mathrm{ltrMax}(\pi)$ uniquely determines the set $\mathrm{Des}(\pi)$. Explicitly, for any $1 \le i \le n-1$,

$$i \in \mathrm{Des}(\pi) \iff i \in \mathrm{ltrMax}(\pi) \text{ and } i+1 \notin \mathrm{ltrMax}(\pi).$$

For a permutation $\pi \in S_n$ let

$$\mathrm{ldes}(\pi) := \max\{i \,:\, i \in \mathrm{Des}(\pi)\},$$

be its *last descent*, with $\mathrm{ldes}(\pi) := 0$ if $\mathrm{Des}(\pi) = \varnothing$ (i.e., if π is the identity permutation).

For a skew shape λ/μ, let *height*(λ/μ) be the number of rows in λ/μ and let $\mathrm{SYT}(\lambda/\mu)$ be the set of standard Young tableaux of shape λ/μ. We use the English convention, according to which row indices increase from top to bottom (see, e.g., [12, Ch. 2.5]). The *descent set* of a standard Young tableau T is

$$\mathrm{Des}(T) := \{i \,:\, i+1 \text{ appears in a lower row of } T \text{ than } i\}.$$

Its *last descent* is

$$\mathrm{ldes}(T) := \max\{i \,:\, i \in \mathrm{Des}(T)\},$$

with $\mathrm{ldes}(T) := 0$ if $\mathrm{Des}(T) = \varnothing$.

We shall make use of the Robinson-Schensted-Knuth (RSK) correspondence which maps each permutation $\pi \in \mathcal{S}_n$ to a pair (P_π, Q_π) of standard Young tableaux of the same shape $\lambda \vdash n$. A detailed description can be found, for example, in [12, Ch. 3.1] or in [15, Ch. 7.11]. A fundamental property of the RSK correspondence is:

Fact 2.3. For each $\pi \in \mathcal{S}_n$, $\mathrm{Des}(P_\pi) = \mathrm{Des}(\pi^{-1})$ and $\mathrm{Des}(Q_\pi) = \mathrm{Des}(\pi)$.

2.2 The k-fold Catalan number

Recall the n-th Catalan number, defined by

$$C_n := \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \qquad (n \ge 0),$$

with generating function

$$c(x) := \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

For each $0 \le k \le n$, the *n-th k-fold Catalan number* $C_{n,k}$ is the coefficient of x^n in $(xc(x))^k$. These numbers are also called *ballot numbers*, and form the *Catalan triangle* [14, A009766]. As proved by Catalan himself [6], they are given explicitly by

$$C_{n,k} = \frac{k}{2n-k}\binom{2n-k}{n} = \binom{2n-k-1}{n-1} - \binom{2n-k-1}{n} \qquad (1 \le k \le n)$$

and $C_{n,0} = \delta_{n,0}$ $(n \ge 0)$; in particular, $C_{n,1} = C_{n-1}$ for $n \ge 1$.

Among the many interpretations of $C_{n,k}$ one can mention the number of lattice paths from $(k,1)$ to (n,n), consisting of steps $(1,0)$ and $(0,1)$, which never go strictly above the line $y = x$; see, e.g., [18, Cor. 16] which uses a slightly different indexing.

The following proposition, reformulating results presented in [7, 18], relates the k-fold Catalan numbers to 321-avoiding permutations and to standard Young tableaux.

Proposition 2.4. ([7, 18]) *For positive integers $1 \le k \le n$,*

$$|\{\pi \in \mathcal{S}_n(321) \,:\, \mathrm{ldes}(\pi^{-1}) = n-k\}| = |SYT(n-1, n-k)| = C_{n,k}.$$

2.3 Quasi-symmetric functions

Schur functions $\{s_\lambda : \lambda \vdash n\}$, indexed by partitions of n, form a distinguished basis for the vector space Λ^n of symmetric functions which are homogeneous of degree n; see, e.g., [15, Corollary 7.10.6]. Recall that a symmetric function in Λ^n is *Schur-positive* if all the coefficients in its expansion in this basis of Schur functions are nonnegative.

Definition 2.5. [15, p. 7.19] A *quasi-symmetric function* (with rational coefficients) in the variables $x = (x_1, x_2, \ldots)$ is a formal power series $f(x) \in \mathbb{Q}[[x]]$, of bounded degree, such that for any nonnegative integer exponents $a_1, \ldots, a_k$ and any two increasing lists of indices $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$

$$[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}]f.$$

Clearly, every symmetric function is quasi-symmetric, but not conversely, as the series $\sum_{i<j} x_i^2 x_j$ shows.

For each subset $D \subseteq [n-1]$ define the *fundamental quasi-symmetric function*

$$\mathcal{F}_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Let $\mathcal{B}$ be a (multi)set of combinatorial objects, equipped with a *descent map* $\mathrm{Des} : \mathcal{B} \to 2^{[n-1]}$ which associates to each element $b \in \mathcal{B}$ a set $\mathrm{Des}(b) \subseteq [n-1]$. Define the quasi-symmetric function

$$\mathcal{Q}(\mathcal{B}) := \sum_{b \in \mathcal{B}} m(b, \mathcal{B}) \mathcal{F}_{n,\mathrm{Des}(b)},$$

where $m(b, \mathcal{B})$ is the multiplicity of the element b in $\mathcal{B}$. With some abuse of terminology, we say that $\mathcal{B}$ is Schur-positive when $\mathcal{Q}(\mathcal{B})$ is.

The following key theorem is due to Gessel.

Proposition 2.6. [15, Theorem 7.19.7] *For every shape $\lambda \vdash n$,*

$$\mathcal{Q}(\mathrm{SYT}(\lambda)) = s_\lambda.$$

Recall the notation P_π from Section 2.1. For every standard Young tableau T of size n, the set

$$\mathcal{C}_T := \{\pi \in \mathcal{S}_n : P_\pi = T\}$$

is the *Knuth class* corresponding to T. Fact 2.3 and Proposition 2.6 imply the following well-known result.

Proposition 2.7. *Knuth classes are Schur-positive.*

The following lemma will be used in Section 5.

Lemma 2.8. [8, Lemma 8.1] *For every Schur-positive set $A \subseteq \mathcal{S}_n$, the (set) statistics* Des *and* $n - \mathrm{Des} := \{n - i \,|\, i \in \mathrm{Des}\}$ *are equi-distributed over A.*

3 The block number of a permutation

3.1 Definitions

Direct sums and block decomposition of permutations appear naturally in the study of pattern-avoiding classes [3, 4].

Let $\pi \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$. The *direct sum* of π and σ is the permutation $\pi \oplus \sigma \in \mathcal{S}_{m+n}$ defined by

$$\pi \oplus \sigma := \begin{cases} \pi(i), & \text{if } i \le m; \\ \sigma(i-m)+m, & \text{otherwise.} \end{cases}$$

For example, if $\pi = 312$ and $\sigma = 2413$ then $\pi \oplus \sigma = 3125746$; see Figure 1.

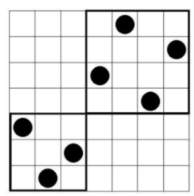

Figure 1: The permutation $312 \oplus 2413 = 3125746$

A nonempty permutation which is not the direct sum of two nonempty permutations is called $\oplus$-*irreducible*. Each permutation π can be written uniquely as a direct sum of $\oplus$-irreducible ones, called the *blocks* of π; their number, denoted by $\mathrm{bl}(\pi)$, is the *block number* of π.

Example 3.1. $\mathrm{bl}(45321) = 1$, $\mathrm{bl}(31254) = 2$, and $\mathrm{bl}(1234) = 4$.

Observation 3.2. The block number of a permutation $\pi \in \mathcal{S}_n$ is

$$\mathrm{bl}(\pi) = 1 + |\{1 \le i \le n-1 \,:\, \max(\pi(1), \ldots, \pi(i)) < \min(\pi(i+1), \ldots, \pi(n))\}|.$$

The following equivalent definition was proposed by Michael Joseph and Tom Roby [11].

Observation 3.3.

$$\mathrm{bl}(\pi) = |\{1 \le i \le n \,:\, (\forall j \le i)\, \pi(j) \le i\}|.$$

3.2 Counting 321-avoiding permutations by block number

Recall from Section 2.2 the Catalan generating function

$$c(x) := \sum_{n=0}^{\infty} C_n x^n.$$

One can prove without difficulty (see [1])

Proposition 3.4. *For any fixed positive integer k, the ordinary generating function for the number of 321-avoiding permutations in $\mathcal{S}_n$ $(n \geq k)$ with exactly k blocks is $(xc(x))^k$.*

Combining Proposition 3.4 with Proposition 2.4 we deduce

Corollary 3.5. *For any integers $1 \leq k \leq n$,*

$$\begin{aligned} |\{\pi \in \mathcal{S}_n(321) : \mathrm{bl}(\pi) = k\}| &= \{\pi \in \mathcal{S}_n(321) : \mathrm{ldes}(\pi^{-1}) = n-k\}| \\ &= |\mathrm{SYT}(n-1, n-k)|. \end{aligned}$$

Corollary 3.5 is refined in this paper; see Theorem 1.3 above and Corollary 4.6 below.

4 Equi-distribution

Definition 4.1. For $1 \leq k \leq n$ denote

$$Bl_{n,k} := \{\pi \in \mathcal{S}_n(321) : \ \mathrm{bl}(\pi) = k\}$$

and

$$L_{n,k} = \{\pi \in \mathcal{S}_n(321) : \ \mathrm{ldes}(\pi^{-1}) = k\}.$$

Theorem 1.3 is proved via a left-to-right-maxima-preserving bijection from $Bl_{n,k}$ to $L_{n,n-k}$.

Definition 4.2. Define maps $f_n : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$, recursively, for all $n \geq 1$. For $n = 1$ the definition is obvious, since $\mathcal{S}_1(321)$ consists of a unique permutation. For $\pi \in \mathcal{S}_n(321)$, $n \geq 2$, the recursive definition of $f_n(\pi)$ depends on $k := \mathrm{bl}(\pi)$ and on the locations of the letters $n-1$ and n in π. Distinguish the following three cases:

Case A: $\pi^{-1}(n) = n$, i.e., n is in the last position.
Then: delete n, apply f_{n-1}, and insert n at the last position.

Case B: $\pi^{-1}(n-1) < \pi^{-1}(n) < n$, i.e., n is to the right of $n-1$ but not in the last position.
Then: delete n, apply f_{n-1}, insert n at the same position as in π, and multiply on the left by the transposition $(n-k-1, n-k)$.

Case C: $\pi^{-1}(n) < \pi^{-1}(n-1)$, i.e., $n-1$ is to the right of n (and must be the last letter, since π is 321-avoiding).
Then: let $\pi' := (n-1, n)\pi$, define $f_n(\pi')$ according to case A above, and multiply it on the left by the cycle $(n-k, n-k+1, ..., n)$.

Remark 4.3. *This recursive definition yields a sequence of permutations $(\pi_n, \pi_{n-1}, \ldots, \pi_1)$, starting with $\pi_n = \pi$. For each $2 \leq i \leq n$, $\pi_{i-1} \in \mathcal{S}_{i-1}$ is obtained from $\pi_i \in \mathcal{S}_i$ by deleting i from π_i (in cases A and B) or by deleting i from $(i-1, i)\pi_i$ (in case C). To recover $f_i(\pi_i)$ from $f_{i-1}(\pi_{i-1})$, the letter i is inserted exactly where it was deleted (for example — in the last position, in cases A and C), and then the permutation is multiplied, on the left, by a suitable cycle.*

Example 4.4. Let $\pi = 31254786 \in \mathcal{S}_8$, so that $\mathrm{bl}(\pi) = 3$ and $\mathrm{ltrMax}(\pi) = \{1,4,6,7\}$. The recursive process is illustrated by the following diagram, where the arrow $\pi_i \to \pi_{i-1}$ is decorated by the case and by the corresponding cycle.

$$\begin{aligned} \pi = \pi_8 = 31254786 \quad & \xrightarrow[(45)]{B} \quad \pi_7 = 3125476 \xrightarrow[(4567)]{C} \pi_6 = 312546 \\ & \xrightarrow{A} \quad \pi_5 = 31254 \xrightarrow[(345)]{C} \pi_4 = 3124 \xrightarrow{A} \pi_3 = 312 \\ & \xrightarrow[(23)]{C} \quad \pi_2 = 21 \xrightarrow[(12)]{C} \pi_1 = 1. \end{aligned}$$

$$\begin{aligned} f_1(\pi_1) = 1 \quad & \xrightarrow{(12)} \quad f_2(\pi_2) = 21 \xrightarrow{(23)} f_3(\pi_3) = 312 \longrightarrow f_4(\pi_4) = 3124 \\ & \xrightarrow{(345)} \quad f_5(\pi_5) = 41253 \longrightarrow f_6(\pi_6) = 412536 \\ & \xrightarrow{(4567)} \quad f_7(\pi_7) = 5126374 \xrightarrow{(45)} f_8(\pi) = f_8(\pi_8) = 41263785. \end{aligned}$$

Note that here $\mathrm{ldes}(f_8(\pi)^{-1}) = 5 = 8 - \mathrm{bl}(\pi)$ and $\mathrm{ltrMax}(f_8(\pi)) = \{1,4,6,7\} = \mathrm{ltrMax}(\pi)$.

Our main claim is

Theorem 4.5. *For each $1 \le k \le n$, the map f_n defined above is a left-to-right-maxima-preserving bijection from $Bl_{n,k}$ onto $L_{n,n-k}$.*

The (quite technical) proof of Theorem 4.5 is given in the full paper version [1].

Theorem 1.3 follows from Theorem 4.5, and implies in turn

Corollary 4.6. *For every positive integer n,*

$$\sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{Des}(\pi)} t^{\pi^{-1}(n)} q^{\mathrm{bl}(\pi)} = \sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{Des}(\pi)} t^{\pi^{-1}(n)} q^{n-\mathrm{ldes}(\pi^{-1})}.$$

Proof. By Observation 2.2, the set $\mathrm{ltrMax}(\pi)$ determines $\mathrm{Des}(\pi)$. In addition, $\pi^{-1}(n)$ is the maximal element of $\mathrm{ltrMax}(\pi)$. □

5 Proof of Theorem 1.2

Setting $t = 1$ in Corollary 4.6 gives

$$\sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{Des}(\pi)} q^{\mathrm{bl}(\pi)} = \sum_{\pi \in \mathcal{S}_n(321)} \mathbf{x}^{\mathrm{Des}(\pi)} q^{n-\mathrm{ldes}(\pi^{-1})},$$

and comparing the coefficients of q^k on both sides gives

$$\mathcal{Q}(Bl_{n,k}) = \mathcal{Q}(L_{n,n-k}) \qquad (1 \le k \le n).$$

It therefore suffices to prove the claim for $L_{n,n-k}$ instead of $Bl_{n,k}$.

Note that a permutation π is 321-avoiding if and only if $height(P_\pi) < 3$, in the terminology of Section 2.1. The set

$$\begin{aligned} L_{n,n-k} &= \{\pi \in \mathcal{S}_n(321) : \mathrm{ldes}(\pi^{-1}) = n-k\} \\ &= \{\pi \in \mathcal{S}_n : height(P_\pi) < 3 \text{ and } \mathrm{ldes}(P_\pi) = n-k\} \end{aligned}$$

is therefore a disjoint union of Knuth classes and thus, by Proposition 2.7, Schur-positive. By the above description of $L_{n,n-k}$, Fact 2.3 and Proposition 2.6,

$$\begin{aligned} \mathcal{Q}(L_{n,n-k}) &= \sum_{\pi \in L_{n,n-k}} \mathcal{F}_{n,\mathrm{Des}(\pi)} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)<3}} \sum_{\substack{P \in \mathrm{SYT}(\lambda) \\ \mathrm{ldes}(P)=n-k}} \sum_{Q \in \mathrm{SYT}(\lambda)} \mathcal{F}_{n,\mathrm{Des}(Q)} \\ &= \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)<3}} |\{P \in \mathrm{SYT}(\lambda) : \mathrm{ldes}(P) = n-k\}| s_\lambda . \end{aligned}$$

We conclude that, for every $\lambda \vdash n$,

$$\langle \mathcal{Q}(Bl_{n,k}), s_\lambda \rangle = \langle \mathcal{Q}(L_{n,n-k}), s_\lambda \rangle = \begin{cases} |\{P \in \mathrm{SYT}(\lambda) : \mathrm{ldes}(P) = n-k\}|, & \text{if } \ell(\lambda) < 3; \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

For $k = n$ we have $\mathrm{ldes}(P) = 0$ only for the unique tableau of shape $\lambda = (n)$, so that

$$\langle \mathcal{Q}(L_{n,0}), s_\lambda \rangle = \delta_{\lambda,(n)}$$

and therefore

$$\mathcal{Q}(L_{n,0}) = s_{(n)}$$

as claimed.

Assume now that $1 \le k \le n-1$, so that $n \le (n-1) + (n-k)$. By the Branching Rule [12, Theorem 2.8.3], if $\lambda \vdash n$ is not contained in $(n-1, n-k)$ then

$$\langle \chi^{(n-1,n-k)} \downarrow^{\mathcal{S}_{2n-k-1}}_{\mathcal{S}_n}, \chi^\lambda \rangle = 0,$$

and also

$$\langle \mathcal{Q}(L_{n,n-k}), s_\lambda \rangle = 0$$

since a standard tableau $P \in \mathrm{SYT}(\lambda)$ (with $\ell(\lambda) < 3$, namely $\lambda = (n-m, m)$ with either $m = 0$ or $m > n-k$) cannot have $\mathrm{ldes}(P) = n-k$.

If $\lambda = (n-m, m)$ with $1 \le m \le n-k$ then

$$\langle \chi^{(n-1,n-k)} \downarrow_{\mathcal{S}_n}^{\mathcal{S}_{2n-k-1}}, \chi^{(n-m,m)} \rangle = |\mathrm{SYT}((n-1, n-k)/(n-m, m))|.$$

Rotating the shape by 180° within a $2 \times n$ box, the right side is seen to be equal to

$$|\mathrm{SYT}((n-m, m)/(k, 1))|.$$

This, in turn, is equal to the number of SYT of shape $(n-m, m)$ with the smallest $k+1$ entries filling a $(k, 1)$ shape in some specific order, say the one with $1, \ldots, k$ in the first row and $k+1$ in the second. These are exactly the SYT of shape $(n-m, m)$ with first descent equal to k. By Lemma 2.8, this number is equal to the number of SYT of shape $(n-m, m)$ with last descent equal to $n-k$, namely to

$$|\{P \in \mathrm{SYT}(\lambda) : \mathrm{ldes}(P) = n-k\}| = \langle \mathcal{Q}(L_{n,n-k}), s_\lambda \rangle.$$

Thus

$$\mathcal{Q}(L_{n,n-k}) = \mathrm{ch}(\chi^{(n-1,n-k)} \downarrow_{\mathcal{S}_n}^{\mathcal{S}_{2n-k-1}}),$$

as claimed. □

6 Final remarks

6.1 Hilbert series

Let P_n/I_n be the quotient of the polynomial ring $P_n = \mathbb{Q}[x_1, \ldots, x_n]$ by the ideal generated by quasi-symmetric functions without constant term. This algebra was studied by Aval, Bergeron and Bergeron [5], who determined its Hilbert series with respect to the grading by total degree in terms of statistics on Dyck paths. An alternative description follows from Corollary 4.6.

Proposition 6.1. *The Hilbert series of the quotient P_n/I_n, graded by total degree, is equal to*

$$\sum_{\pi \in \mathcal{S}_n(321)} q^{n-\mathrm{bl}(\pi)}.$$

It is now desired to find two different bases for the graded ring P_n/I_n, both indexed by 321-avoiding permutations, with total degree equal to the block-number and to the last descent, respectively. Determining a nicely behaved linear action of the symmetric group (or of the Temperley-Lieb algebra) on these bases may provide a representation theoretic proof of Corollary 4.6.

6.2 Schur-positive pattern-statistic pairs

Sagan and Woo [13] raised the problem of finding Schur-positive pattern-avoiding sets. A natural goal is to look, further, for Schur-positive statistics on pattern-avoiding sets.

Definition 6.2. Let $\mathrm{stat}: \mathcal{S}_n \longrightarrow \mathbb{N}$ be a permutation statistic, and let $\varnothing \neq \Pi \subseteq \mathcal{S}_m$ be a nonempty set of patterns. The *pattern-statistic pair* (Π, stat) is *Schur-positive* if

$$\mathcal{Q}(\{\pi \in \mathcal{S}_n(\Pi) : \mathrm{stat}(\pi) = k\})$$

is Schur-positive for all integers $n \geq 1$ and $k \geq 0$.

By Proposition 2.7, sets of permutations which are closed under Knuth relations are Schur-positive. It follows that if $\mathcal{S}_n(\Pi)$ is closed under Knuth relations for every n, and the statistic stat is also invariant under these relations, then (Π, stat) is a Schur-positive pair. For example, letting $\mathrm{ides}(\pi) := |\mathrm{Des}(\pi^{-1})|$ and $e_m \in \mathcal{S}_m$ be the identity permutation, the pair $(\{e_m\}, \mathrm{ides})$ is Schur-positive. For similar reasons, the pair $(\{132, 312\}, \mathrm{imaj})$ is Schur-positive, where $\mathrm{imaj}(\pi)$ is the major index of π^{-1}.

An example of a different type was given in this paper: By Theorem 1.2, $(\{321\}, \mathrm{bl})$ is a Schur-positive pair. Note that the block number is not invariant under Knuth relations.

Problem 6.3. Find other Schur-positive pattern-statistic pairs, which are not invariant under Knuth relations.

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